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2005 J. Phys. A: Math. Gen. 38 2623

(http://iopscience.iop.org/0305-4470/38/12/006)

J. Phys. A: Math. Gen. **38** (2005) 2623–2633 doi:10.1088/0305-4470/38/12/006

On dispersionless Hirota equations of the dispersionless Dym hierarchy

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Received 30 October 2004, in final form 4 January 2005
Published 9 March 2005
Online at stacks.iop.org/JPhysA/38/2623

Abstract

We show that, under twistor construction, the only finite-dimensional reduction of the dispersionless Dym hierarchy which can be characterized by a free energy is the two-reduction model with Lax operator of the form $\lambda = up + u_0 + p^{-1}$. The primary variables u and u_0 can be expressed in terms of second derivatives of the associated free energy and the hierarchy flows can be written as a set of dispersionless Hirota equations.

PACS number: 02.30.Ik

1. Introduction

An important class of dispersionless integrable hierarchies (see [1] for a review) is the one which can be formulated in terms of Laurent series of the form $\Lambda = \sum_i^n a_i p^i$ with respect to the Poisson bracket $\{\,,\}$ defined by

$$\{f(x,p), g(x,p)\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}. \tag{1.1}$$

Denoting the projections by $\Lambda_{\geqslant k} = \sum_{i\geqslant k} a_i p^i$ and $\Lambda_{<k} = \sum_{i<k} a_i p^i$ then from Lie–Poisson algebra point of view [2], the decomposition $\Lambda = \Lambda_{\geqslant k} \oplus \Lambda_{<k}$ with respect to the Poisson bracket (1.1) is a Lie subalgebra decomposition only for $k = 0, 1, 2$, namely $\{\Lambda_{\geqslant k}, \Lambda_{\geqslant k}\} \subseteq \Lambda_{\geqslant k}$ and $\{\Lambda_{<k}, \Lambda_{<k}\} \subseteq \Lambda_{<k}$. Consequently, one can introduce dispersionless Lax hierarchies according to the above decompositions as

$$\frac{\partial \lambda}{\partial t_i} = \{(\lambda^{i/n})_{\geqslant k}, \lambda\}, \qquad \lambda \in \Lambda \tag{1.2}$$

which provides a system of evolution equations on the coefficients of λ for each of the following three classes: the dispersionless Kadomtsev–Petviashvili (dKP) hierarchy [3–5] for $k = 0$, the dispersionless modified Kadomtsev–Petviashvili (dmKP) hierarchy [6, 7] for $k = 1$

and the dispersionless Dym (dDym) hierarchy [8] for $k = 2$. Among them we shall focus on the dDym hierarchy since its integrability properties are less investigated. Although some progresses on the dDym hierarchy have been made in the past few years, such as bi-Hamiltonian structure, Miura map, hodograph solutions, additional symmetries and twistor construction, etc (see e.g. [8–14]), the existence problem of the primary free energy (free energy for short) for the dDym hierarchy has not been explored. In the context of integrable hierarchies/2d topological field theory correspondences, the dispersionless limit of the logarithm of the τ-function corresponds to the free energy of some topological field theory. In particular, the coefficients of the expansion of the free energy in flat coordinates provide the values of the correlation functions at genus zero (see, e.g., [5,15–21] and references therein). Also the free energy is governed by a set of equations called dispersionless Hirota equations [1, 22, 23] that are intimately related to the Witten–Dijkgraff–Verlinde–Verlinde (WDVV) equations of associativity [20, 24]. Thus it is an important issue for investigating those dDym systems that can be characterized by a free energy and to study their underlying topological field theories.

In this work, we attempt to address this problem for finite-dimensional reductions of the dDym hierarchy through the analysis of S function which can be viewed as the WKB phase function of the Baker–Akhiezer function [3]. It was also introduced by Krichever in the context of topological field theories for the dKP hierarchy [5] and then elaborated by Takasaki and Takebe in the twistor construction for the cases of dKP and dToda [1, 25–27]. Using twistor construction we shall show that the only finite-dimensional reduction of the dDym hierarchy which can be characterized by a free energy is the two-reduction model with Lax operator of the form $\lambda = up + u_0 + p^{-1}$. A peculiar property of the two-reduction model is that, besides the Lax equations (1.2), it also contains an extra flow generated by the Hamiltonian $\mathcal{B}_0 = (\log\lambda)_{\geqslant 2}$, resembling the log flow introduced by Krichever in the construction of the Whitham hierarchy [17]. Due to this extra flow, it turns out that the primary variables u and u_0 can be expressed in terms of second derivatives of the free energy and the hierarchy flows can be written as a set of dispersionless Hirota equations.

Our presentation is organized as follows. In section 2, after introducing the Lax form of the dDym hierarchy we re-formulate it in a dToda-like fashion to incorporate finite-dimensional reductions. In section 3, we briefly recall the dressing operator approach for the Lax operators and associated Orlov–Schulman [1, 28] operators. The twistor construction for solutions of the dDym hierarchy is also given. In sections 4 and 5 we show that the S functions introduced in the twistor construction can be characterized by a free energy only for the two-reduction model which has a Lax operator of the form $\lambda = up + u_0 + p^{-1}$. In section 6, we derive the dispersionless Hirota equations of the two-reduction model. Concluding remarks are presented in section 7.

2. Extended dDym hierarchy

The dDym hierarchy [8] (see also [9, 10]) is defined by the Lax equation

$$\frac{\partial\lambda}{\partial t_n} = \{\mathcal{B}_n, \lambda\}, \qquad \mathcal{B}_n = (\lambda^n)_{\geqslant 2}, \tag{2.1}$$

where the Lax operator λ is a Laurent series of the form

$$\lambda = up + u_0 + u_1 p^{-1} + u_2 p^{-2} + \cdots.$$

Since $\mathcal{B}_1 = 0$, u_i do not depend on t_1. The first member of the Lax flows (2.1) is the so-called dDym equation in $(2 + 1)$-dimensions

$$u_t = \frac{3}{4}\frac{1}{u}\left[u^2\partial_x^{-1}\left(\frac{u_y}{u^2}\right)\right]_y, \tag{2.2}$$

where we denote $t_2 = y$ and $t_3 = t$. Equation (2.2) is just the dispersionless limit of the ordinary (2 +1)-dimensional Dym equation [29] by dropping the dispersion term.

To study finite-dimensional reductions and being motivated by the dispersionless Toda (dToda) system [1, 25, 27], we formulate the dDym hierarchy in terms of two Laurent series λ and $\hat{\lambda}$ of the form:

$$\lambda = up + \sum_{n=0}^{\infty} u_n p^{-n}, \qquad \hat{\lambda}^{-1} = p^{-1} + \sum_{n=0}^{\infty} \hat{u}_n p^n, \tag{2.3}$$

that obey the Lax equations

$$\begin{aligned} \frac{\partial\lambda}{\partial t_n} &= \{\mathcal{B}_n, \lambda\}, & \frac{\partial\hat{\lambda}}{\partial t_n} &= \{\mathcal{B}_n, \hat{\lambda}\}, \\ \frac{\partial\lambda}{\partial \hat{t}_n} &= \{\hat{\mathcal{B}}_n, \lambda\}, & \frac{\partial\hat{\lambda}}{\partial \hat{t}_n} &= \{\hat{\mathcal{B}}_n, \hat{\lambda}\}, \end{aligned} \tag{2.4}$$

where $\mathcal{B}_n = (\lambda^n)_{\geqslant 2}, \hat{\mathcal{B}}_n = (\hat{\lambda}^{-n})_{\leqslant 1}$ and the coefficient functions u_n and $\hat{u}_n$ depend on the variables $(x, t_2, t_3, \ldots)$ and $(\hat{t}_{-1}, \hat{t}_1, \ldots)$. In particular, since $\hat{\mathcal{B}}_{-1} = p$, u_n and $\hat{u}_n$ depend on $\hat{t}_{-1}$ and x only through the linear combination $\hat{t}_{-1} + x$ and we may simply put $\hat{t}_{-1} = x$.

The Lax equations of λ and $\hat{\lambda}$ are equivalent to the zero-curvature equations

$$\begin{aligned} &\frac{\partial\mathcal{B}_m}{\partial t_n} - \frac{\partial\mathcal{B}_n}{\partial t_m} + \{\mathcal{B}_m, \mathcal{B}_n\} = 0, \\ &\frac{\partial\hat{\mathcal{B}}_m}{\partial \hat{t}_n} - \frac{\partial\hat{\mathcal{B}}_n}{\partial \hat{t}_m} + \{\hat{\mathcal{B}}_m, \hat{\mathcal{B}}_n\} = 0, \\ &\frac{\partial\mathcal{B}_m}{\partial \hat{t}_n} - \frac{\partial\hat{\mathcal{B}}_n}{\partial t_m} + \{\mathcal{B}_m, \hat{\mathcal{B}}_n\} = 0, \end{aligned} \tag{2.5}$$

that guarantee the commutativity of the Lax equations (2.4). Since (2.4) is an extension of the dDym hierarchy by introducing the time parameters $\hat{t}_n$, we call (2.4) the extended dDym (EdDym) hierarchy. In addition to the $(2+1)$-dimensional dDym equation (2.2) which corresponds to the first equation of (2.5) for $n = 2, m = 3$, the simplest $(2+1)$-dimensional equation involving $\hat{t}_n$ is given by the third equation for $n = 1, m = 2$ as

$$u_{\hat{t}_1} + u^2\left[2u^{-1}\partial_y^{-1}u^2\left(\partial_y^{-1}u^2\right)_{xx}\right]_x = 0. \tag{2.6}$$

We refer it to the EdDym equation. Furthermore, using the Lax equations (2.4) and the fact that $\int \mathrm{res}\{A, B\} = 0$ for any two Laurent series A and B, one can verify easily that the EdDym hierarchy equips an infinite number of conserved quantities defined by

$$H_n = \frac{1}{n}\int \mathrm{res}(\lambda^n\,\mathrm{d}p), \qquad \hat{H}_n = \frac{1}{n}\int \mathrm{res}(\hat{\lambda}^{-n}\,\mathrm{d}p), \qquad n = 1, 2, \ldots$$

where we denote the residue of Λ as $\mathrm{res}(\Lambda\,\mathrm{d}p) = a_{-1}$.

Remark. In [13] an interesting generalization called r-dToda hierarchy ($r \in Z$) was formulated, in which the Lax equations are defined by the generalized Poisson bracket $\{\Lambda_1, \Lambda_2\} = p^r(\partial\Lambda_1/\partial p\partial\Lambda_2/\partial x - \partial\Lambda_1/\partial x\partial\Lambda_2/\partial p)$ [9] with respect to the subalgebra decomposition $\Lambda = \Lambda_{\geqslant 1-r} \oplus \Lambda_{<1-r}$. As $r = 0$ the above generalized Poisson bracket reduces to (1.1); however, the corresponding dToda-type system is similar to but different from the EdDym system due to different Lie algebraic splittings of Λ.

3. Dressing formulation and twistor construction

Let $(\lambda, \hat{\lambda})$ be a solution of the Lax equations (2.4) and has the form (2.3). Then there exist Laurent series $\Theta = \sum_{n=-1}^{\infty} \Theta_n(x, t, \hat{t}) p^{-n}$ and $\hat{\Theta} = \sum_{n=2}^{\infty} \hat{\Theta}_n(x, t, \hat{t}) p^n$ such that

$$\lambda = \mathrm{e}^{\mathrm{ad}\Theta}(p), \qquad \hat{\lambda} = \mathrm{e}^{\mathrm{ad}\hat{\Theta}}(p) \tag{3.1}$$

with dressing functions Θ and $\hat{\Theta}$ defined by

$$\begin{aligned} \nabla_{t_n,\Theta}\Theta &= -(\lambda^n)_{\leqslant 1}, \qquad & \nabla_{t_n,\hat{\Theta}}\hat{\Theta} &= (\lambda^n)_{\geqslant 2}, \\ \nabla_{\hat{t}_n,\Theta}\Theta &= (\hat{\lambda}^{-n})_{\leqslant 1}, \qquad & \nabla_{\hat{t}_n,\hat{\Theta}}\hat{\Theta} &= -(\hat{\lambda}^{-n})_{\geqslant 2}, \end{aligned} \tag{3.2}$$

where $\mathrm{ad}A(B) = \{A, B\}$ and $\nabla_{t_n,A} B \equiv \sum_{k=0}^{\infty} (\mathrm{ad}A)^k \partial_{t_n} B/(k+1)!$. Using the formula $\partial_n \mathrm{e}^{\mathrm{ad}A}(B) = \mathrm{e}^{\mathrm{ad}A}(\partial_n B) + \{\nabla_{t_n,A} A, \mathrm{e}^{\mathrm{ad}A}(B)\}$ [1] and (3.2) it is easy to show that the dressed operators λ and $\hat{\lambda}$ satisfy the Lax equations (2.4).

Given the dressing form (3.1), one can also introduce the associated Orlov–Schulman operators [1, 28] $(\mathcal{M}, \hat{\mathcal{M}})$ as

$$\mathcal{M} = \mathrm{e}^{\mathrm{ad}\Theta}\left(x + \sum_{n=2}^{\infty} n t_n p^{n-1}\right) = \mathrm{e}^{\mathrm{ad}\Theta}\, \mathrm{e}^{\mathrm{ad}t(p)}(x),$$

$$\hat{\mathcal{M}} = \mathrm{e}^{\mathrm{ad}\hat{\Theta}}\left(x - \sum_{n=1}^{\infty} n \hat{t}_n p^{-n-1}\right) = \mathrm{e}^{\mathrm{ad}\hat{\Theta}}\, \mathrm{e}^{\mathrm{ad}\hat{t}(p)}(x),$$

where $t(p) = \sum_{n=2}^{\infty} t_n p^n$ and $\hat{t}(p) = \sum_{n=1}^{\infty} \hat{t}_n p^{-n}$. They have the following expansions,

$$\mathcal{M} = \sum_{n=2}^{\infty} n t_n \lambda^{n-1} + \int^x \frac{1}{u} + \sum_{n=1} v_n \lambda^{-n-1},$$

$$\hat{\mathcal{M}} = -\sum_{n=1}^{\infty} n \hat{t}_n \hat{\lambda}^{-n-1} + x + \sum_{n=2} \hat{v}_n \hat{\lambda}^{n-1},$$

where v_n and $\hat{v}_n$ are also functions of x, t_n and $\hat{t}_n$ and satisfy the Lax equations

$$\begin{aligned} \frac{\partial \mathcal{M}}{\partial t_n} &= \{\mathcal{B}_n, \mathcal{M}\}, \qquad & \frac{\partial \hat{\mathcal{M}}}{\partial t_n} &= \{\mathcal{B}_n, \hat{\mathcal{M}}\}, \\ \frac{\partial \mathcal{M}}{\partial \hat{t}_n} &= \{\hat{\mathcal{B}}_n, \mathcal{M}\}, \qquad & \frac{\partial \hat{\mathcal{M}}}{\partial \hat{t}_n} &= \{\hat{\mathcal{B}}_n, \hat{\mathcal{M}}\}, \end{aligned} \tag{3.3}$$

and the canonical relations

$$\{\lambda, \mathcal{M}\} = \{\hat{\lambda}, \hat{\mathcal{M}}\} = 1. \tag{3.4}$$

The integrability of the EdDym hierarchy can be characterized by the 2-form

$$\omega = \mathrm{d}p \wedge \mathrm{d}x + \sum_{n=2} \mathrm{d}\mathcal{B}_n \wedge \mathrm{d}t_n + \sum_{n=1} \mathrm{d}\hat{\mathcal{B}}_n \wedge \mathrm{d}\hat{t}_n \tag{3.5}$$

which is closed $\mathrm{d}\omega = 0$, and $\omega \wedge \omega = 0$ implies the zero-curvature equations (2.5). The conjugate pairs $(\lambda, \mathcal{M})$ and $(\hat{\lambda}, \hat{\mathcal{M}})$ thus provide the Darboux coordinates in a neighbourhood of $p = \infty$ and $p = 0$, respectively. In the overlapping domain, the two Darboux coordinates are related by

$$\omega = \mathrm{d}\lambda \wedge \mathrm{d}\mathcal{M} = \mathrm{d}\hat{\lambda} \wedge \mathrm{d}\hat{\mathcal{M}} \tag{3.6}$$

which yields the Lax equations (2.4), (3.3) and the canonical relation (3.4).

Having established the integrability structure of the EdDym hierarchy we can give a construction of solutions based on the twistor construction (or Riemann–Hilbert problem) similar to that of the dToda hierarchy [1]. Let $(\lambda, \mathcal{M})$ and $(\hat{\lambda}, \hat{\mathcal{M}})$ have the form defined above. Then it can be shown [12] that if the given functions $f(p, x)$, $g(p, x)$, $\hat{f}(p, x)$ and $\hat{g}(p, x)$ satisfy

$$\{f(p, x), g(p, x)\} = 1, \qquad \{\hat{f}(p, x), \hat{g}(p, x)\} = 1,$$

then the functional equations

$$f(\lambda, \mathcal{M}) = \hat{f}(\hat{\lambda}, \hat{\mathcal{M}}), \qquad g(\lambda, \mathcal{M}) = \hat{g}(\hat{\lambda}, \hat{\mathcal{M}})$$

give a solution $(\lambda, \mathcal{M}, \hat{\lambda}, \hat{\mathcal{M}})$ which satisfies (2.4), (3.3), and the canonical relation (3.4). We call $(f, g, \hat{f}, \hat{g})$ the twistor data of the associated solution. In fact, besides the solution structure the twistor-theoretic construction has been used to study additional symmetries and conformal map of some dispersionless integrable hierarchies (see, e.g., [1, 30, 31]).

4. *S* function and twistor reduction

From expressions (3.5) and (3.6) of the 2-form ω there exist functions S and $\hat{S}$ of the form

$$S = \sum_{n=2} t_n \lambda^n + \phi_1 \lambda + \phi_0 - \sum_{n=1} \frac{v_n}{n} \lambda^{-n}, \qquad \hat{S} = \sum_{n=1} \hat{t}_n \hat{\lambda}^{-n} + x\hat{\lambda} + \sum_{n=2} \frac{\hat{v}_n}{n} \hat{\lambda}^n \tag{4.1}$$

where $\phi_1 = \int^x 1/u$ and $\phi_0 = -\int^x u_0/u$, such that

$$\mathrm{d}S = \mathcal{M}\,\mathrm{d}\lambda + p\,\mathrm{d}x + \sum_{n=2} \mathcal{B}_n\,\mathrm{d}t_n + \sum_{n=1} \hat{\mathcal{B}}_n\,\mathrm{d}\hat{t}_n,$$
$$\mathrm{d}\hat{S} = \hat{\mathcal{M}}\,\mathrm{d}\hat{\lambda} + p\,\mathrm{d}x + \sum_{n=2} \mathcal{B}_n\,\mathrm{d}t_n + \sum_{n=1} \hat{\mathcal{B}}_n\,\mathrm{d}\hat{t}_n$$

or, equivalently,

$$\begin{aligned} \mathcal{M} &= \left.\frac{\partial S}{\partial \lambda}\right|_{x,t,\hat{t}\text{ fixed}}, & \hat{\mathcal{M}} &= \left.\frac{\partial \hat{S}}{\partial \hat{\lambda}}\right|_{x,t,\hat{t}\text{ fixed}}, \\ p &= \left.\frac{\partial S}{\partial x}\right|_{\lambda,t,\hat{t}\text{fixed}}, & p &= \left.\frac{\partial \hat{S}}{\partial x}\right|_{\hat{\lambda},t,\hat{t}\text{ fixed}}, \end{aligned} \tag{4.2}$$

$$\mathcal{B}_n = \left.\frac{\partial S}{\partial t_n}\right|_{\lambda,x,t_{m\neq n},\hat{t}\text{ fixed}}, \qquad \mathcal{B}_n = \left.\frac{\partial \hat{S}}{\partial t_n}\right|_{\hat{\lambda},x,t_{m\neq n},\hat{t}\text{ fixed}}, \tag{4.3}$$

$$\hat{\mathcal{B}}_n = \left.\frac{\partial S}{\partial \hat{t}_n}\right|_{\lambda,x,t,\hat{t}_{m\neq n}\text{ fixed}}, \qquad \hat{\mathcal{B}}_n = \left.\frac{\partial \hat{S}}{\partial \hat{t}_n}\right|_{\hat{\lambda},x,t,\hat{t}_{m\neq n}\text{ fixed}}. \tag{4.4}$$

These relations can be verified through the following formulae:

$$\begin{aligned} \frac{\partial v_k}{\partial t_n} &= \operatorname{res}(\lambda^k d_p \mathcal{B}_n), & \frac{\partial \hat{v}_k}{\partial t_n} &= \operatorname{res}(\hat{\lambda}^{-k} d_p \mathcal{B}_n), \\ \frac{\partial v_k}{\partial \hat{t}_n} &= \operatorname{res}(\lambda^k d_p \hat{\mathcal{B}}_n), & \frac{\partial \hat{v}_k}{\partial \hat{t}_n} &= \operatorname{res}(\hat{\lambda}^{-k} d_p \hat{\mathcal{B}}_n), \\ \frac{\partial v_k}{\partial x} &= \operatorname{res}(\lambda^k\,\mathrm{d}p), & \frac{\partial \hat{v}_k}{\partial x} &= \operatorname{res}(\hat{\lambda}^{-k}\,\mathrm{d}p). \end{aligned} \tag{4.5}$$

We remark that the compatibility conditions of equations (4.2)–(4.4), namely,

$$\frac{\partial p}{\partial t_n} = \frac{\partial \mathcal{B}_n}{\partial x}, \qquad \frac{\partial p}{\partial \hat{t}_n} = \frac{\partial \hat{\mathcal{B}}_n}{\partial x}$$

are equivalent to the Lax equations (2.4).

Although the integrability of the EdDym hierarchy can be characterized by the functions S and $\hat{S}$ as shown above, however, in contrast to the cases of dKP and dToda, the existence of a free energy (tau function) for the dDym system is quite restricted. The reason is the following. Observing that

$$\operatorname{res}(\lambda^k d_p \lambda^n) = \operatorname{res}(\hat{\lambda}^{-k} d_p \hat{\lambda}^{-n}) = 0, \qquad k, n \geqslant 1$$

which together with (4.5) implies

$$\frac{\partial v_k}{\partial t_n} - \frac{\partial v_n}{\partial t_k} = \lambda^k_{[1]}\lambda^n_{[-1]} - \lambda^k_{[-1]}\lambda^n_{[1]}, \tag{4.6}$$

$$\frac{\partial \hat{v}_k}{\partial \hat{t}_n} - \frac{\partial \hat{v}_n}{\partial \hat{t}_k} = \hat{\lambda}^{-k}_{[-1]}\hat{\lambda}^{-n}_{[1]} - \hat{\lambda}^{-k}_{[1]}\hat{\lambda}^{-n}_{[-1]}, \tag{4.7}$$

where $k, n \geqslant 1$ and $\Lambda_{[k]} = a_k$. The right-hand side of equations (4.6) does not vanish unless the following equations are fulfilled,

$$\frac{\lambda^k_{[1]}}{\lambda^k_{[-1]}} = \frac{\lambda_{[1]}}{\lambda_{[-1]}} = \frac{u}{u_1}, \tag{4.8}$$

where the Hamiltonian densities $\lambda^k_{[-1]} \neq 0$ provided that $uu_0u_1 \neq 0$. Using the formula $\operatorname{res}(\lambda^n d_p \lambda) = 0, n > 0$ and denoting $\lambda^n = \sum_j^n \lambda^n_{[j]} p^j$ we have $0 = \operatorname{res}(\lambda^n d_p \lambda) = u\lambda^n_{[-1]} - u_1\lambda^n_{[1]} - \cdots - nu_n\lambda^n_{[n]}$ which together with (4.8) implies that $2u_2\lambda^n_{[2]} + \cdots + nu_n\lambda^n_{[n]} = 0$ for $n \geqslant 2$ and by induction we have $u_n = 0, n \geqslant 2$. That means $\lambda = up + u_0 + u_1 p^{-1}$. Similarly, the right-hand side of (4.7) vanishes which leads to $\hat{\lambda}^{-1} = p^{-1} + \hat{u}_0 + \hat{u}_1 p$. Now taking into account the twistor condition:

$$\lambda = \hat{\lambda}^{-1} = up + u_0 + p^{-1} \tag{4.9}$$

which identifies $\hat{u}_1 = u, \hat{u}_0 = u_0, u_1 = 1$ and corresponds to the twistor data $(f(p, x) = p, \hat{f}(p, x) = p^{-1})$. As a consequence, the time variables $\hat{t}_n$ can be eliminated via the identification $\hat{t}_n = -t_n$ and (4.9) represents a two-reduction solution for the dDym hierarchy. Therefore, there exists a potential $\mathcal{F}(t, x)$ such that $v_n = \partial\mathcal{F}/\partial t_n$. Note that

$$\frac{\partial \phi_1}{\partial t_n} = -\frac{\partial v_n}{\partial x} = -\frac{\partial^2 \mathcal{F}}{\partial x \partial t_n},$$

which yields $\phi_1 = -\partial\mathcal{F}/\partial x$; however, ϕ_0 defined in (4.1) cannot be expressed as a derivative of $\mathcal{F}$. To solve this problem, inspired by [23, 32–35], we introduce an extra flow so that all variables in S function (and hence in $\mathcal{B}_n$) can be expressed in terms of derivatives of $\mathcal{F}$. We shall see that this extra flow is crucial for obtaining the corresponding dispersionless Hirota equations.

5. Extra flow and free energy

Let us extend the Lax equation (2.1) by introducing an extra parameter t_0 as

$$\frac{\partial \lambda}{\partial t_n} = \{\mathcal{B}_n, \lambda\}, \qquad \frac{\partial \lambda}{\partial t_0} = \{\mathcal{B}_0, \lambda\} \tag{5.1}$$

where $\mathcal{B}_0 = (\log\lambda)_{\geqslant 2}$ and the Lax operator λ is defined by (4.9) with u and u_0 now being functions of t_n, x and t_0. Here we have used the following prescription [32] for $\log\lambda$ to avoid an appearance of $\log p$:

$$\log(up + u_0 + p^{-1}) = \frac{1}{2}\log(u) + \frac{1}{2}\log\left(1 + \frac{u_0}{u}p^{-1} + \frac{1}{u}p^{-2}\right) + \frac{1}{2}\log(1 + u_0 p + up^2)$$

where we shall Taylor expand the second term in p^{-1}, whereas in p for the last term.

The dressing formulation is still valid if we extend the dressing function Θ defined in (3.2) to include t_0 such that

$$\nabla_{t_0,\Theta}\Theta = -(\log\lambda)_{\leqslant 1}.$$

On the other hand, the modified Orlov–Schulman operator $\bar{\mathcal{M}}$ is then defined by

$$\begin{aligned}\bar{\mathcal{M}} &= \mathrm{e}^{\mathrm{ad}\Theta}\left(\sum_{n=2} nt_n p^{n-1} + x + t_0 p^{-1}\right)\\ &= \sum_{n=2}^{\infty} nt_n\lambda^{n-1} + \int^x \frac{1}{u} + t_0\lambda^{-1} + \sum_{n=1} v_n\lambda^{-n-1}\end{aligned}$$

which also satisfies the canonical relation $\{\lambda, \bar{\mathcal{M}}\} = 1$. It turns out that $\bar{\mathcal{M}}$ obeys the Lax flows

$$\frac{\partial\bar{\mathcal{M}}}{\partial t_n} = \{\mathcal{B}_n, \bar{\mathcal{M}}\}, \qquad \frac{\partial\bar{\mathcal{M}}}{\partial t_0} = \{\mathcal{B}_0, \bar{\mathcal{M}}\}.$$

Also the conjugate pair $(\lambda, \bar{\mathcal{M}})$ provides the Darboux coordinate of the associated 2-form $\omega = \mathrm{d}p \wedge \mathrm{d}x + \sum_{n=2} \mathrm{d}\mathcal{B}_n \wedge \mathrm{d}t_n + \mathrm{d}\mathcal{B}_0 \wedge \mathrm{d}t_0 = \mathrm{d}\lambda \wedge \bar{\mathcal{M}}$ which implies the existence of a $\bar{S}$ function such that

$$\mathrm{d}\bar{S} = \bar{\mathcal{M}}\,\mathrm{d}\lambda + p\,\mathrm{d}x + \sum_{n=2}\mathcal{B}_n\,dt_n + \mathcal{B}_0\,\mathrm{d}t_0$$

or, equivalently,

$$\begin{aligned}\bar{\mathcal{M}} &= \left.\frac{\partial\bar{S}}{\partial\lambda}\right|_{x,t,t_0\ \text{fixed}}, & p &= \left.\frac{\partial\bar{S}}{\partial x}\right|_{\lambda,t,t_0\ \text{fixed}},\\ \mathcal{B}_n &= \left.\frac{\partial\bar{S}}{\partial t_n}\right|_{\lambda,x,t_{m\neq n},t_0\ \text{fixed}}, & \mathcal{B}_0 &= \left.\frac{\partial\bar{S}}{\partial t_0}\right|_{\lambda,x,t\ \text{fixed}}.\end{aligned} \tag{5.2}$$

Using the residue formula $\mathrm{res}(\lambda^k d_p\lambda) = \delta_{k,-1}$ and the expansion for $\bar{\mathcal{M}}$ we have

$$\frac{\partial v_k}{\partial x} = \mathrm{res}(\lambda^k\,\mathrm{d}p), \qquad \frac{\partial v_k}{\partial t_n} = \mathrm{res}(\lambda^k d_p\mathcal{B}_n), \qquad \frac{\partial v_k}{\partial t_0} = \mathrm{res}(\lambda^k d_p\mathcal{B}_0) \tag{5.3}$$

which shows that $\bar{S}$ is given by

$$\bar{S} = \sum_{n=2} t_n\lambda^n + \phi_1\lambda + \phi_0 + t_0\log\lambda - \sum_{n=1}\frac{v_n}{n}\lambda^{-n}.$$

Furthermore, from (5.3) it follows that

$$\frac{\partial v_k}{\partial t_n} - \frac{\partial v_n}{\partial t_k} = 0, \qquad \frac{\partial}{\partial t_n}\frac{\partial v_k}{\partial t_0} - \frac{\partial}{\partial t_k}\frac{\partial v_n}{\partial t_0} = 0, \qquad \frac{\partial}{\partial t_n}\frac{\partial v_k}{\partial x} - \frac{\partial}{\partial t_k}\frac{\partial v_n}{\partial x} = 0. \tag{5.4}$$

Hence, from the first equation in (5.4), there exists a function $\mathcal{F}(x, t_0, t)$ which is the extension of the previous $\mathcal{F}$ to include t_0, such that $v_n = \partial\mathcal{F}/\partial t_n$, while by the second and the third there exist potentials v_0 and $\bar{v}$ such that

$$\frac{\partial v_n}{\partial t_0} = \frac{\partial v_0}{\partial t_n}, \qquad \frac{\partial v_n}{\partial x} = \frac{\partial\bar{v}}{\partial t_n},$$

which together with $v_n = \partial\mathcal{F}/\partial t_n$ shows that $v_0 = \partial\mathcal{F}/\partial t_0 = -\phi_0/2$ and $\bar{v} = \partial\mathcal{F}/\partial x = -\phi_1 + t_1$. Hence, the primary variables can be expressed as

$$u = -\frac{1}{\mathcal{F}_{xx}}, \qquad u_0 = -2\frac{\mathcal{F}_{0x}}{\mathcal{F}_{xx}}. \tag{5.5}$$

Now denoting $\partial^2\mathcal{F}/\partial t_n\partial t_m = \mathcal{F}_{nm}$, then p, $\mathcal{B}_n$ and $\mathcal{B}_0$ can be written as

$$\begin{aligned} p(\lambda) &= -\mathcal{F}_{xx}\lambda - 2\mathcal{F}_{x0} - \sum_{m=1}\frac{\mathcal{F}_{xm}}{m}\lambda^{-m}, \\ \mathcal{B}_n(\lambda) &= \lambda^n - \mathcal{F}_{nx}\lambda - 2\mathcal{F}_{n0} - \sum_{m=2}\frac{\mathcal{F}_{nm}}{m}\lambda^{-m}, \\ \mathcal{B}_0(\lambda) &= \log\lambda - \mathcal{F}_{0x}\lambda - 2\mathcal{F}_{00} - \sum_{m=2}\frac{\mathcal{F}_{0m}}{m}\lambda^{-m}. \end{aligned} \tag{5.6}$$

6. Dispersionless Hirota equations

In this section we would like to show that the two-reduction hierarchy flows (5.1) can be characterized by the second derivatives of $\mathcal{F}$ defined by

$$\left(\sum_{n=1}\frac{\lambda^n_{[1]}}{n}\mu^{-n}\right)\frac{p(\lambda)}{p(\mu) - p(\lambda)} = \sum_{n=2}\partial_p Q_n(\lambda)\mu^{-n}, \tag{6.1}$$

where $Q_n \equiv \mathcal{B}_n/n$. In other words, the $\mathcal{F}_{nm}$ defined in (5.6) satisfy a set of dispersionless Hirota equations. To see this, following [22] (see also [1]), let us write (5.2) as

$$p(\lambda) = \frac{\lambda}{u} - \frac{u_0}{u} - \sum_{k=1}\frac{1}{k}\frac{\partial v_k}{\partial x}\lambda^{-k}.$$

Multiplying both sides by $\lambda^{n-1}\partial_p\lambda$, we have

$$\lambda^n\partial_p\lambda = (up(\lambda) + u_0)\lambda^{n-1}\partial_p\lambda + \sum_{j=1} p_{j+1}\lambda^{n-j-1}\partial_p\lambda,$$

where $p_{j+1} \equiv u\partial_x v_j/j$. Taking the projection $(\,)_{\geqslant 1}$ we obtain the recurrence relation

$$\partial_p Q_{n+1}(\lambda) = (up(\lambda) + u_0)\partial_p Q_n(\lambda) + up(\lambda)\frac{\lambda^n_{[1]}}{n} + \sum_{j=1}^{n-2} p_{j+1}\partial_p Q_{n-j}(\lambda). \tag{6.2}$$

Multiplying (6.2) by μ^{-n} and summing over n we obtain

$$\left(\mu - (up(\lambda) + u_0) - \sum_{j=1} p_{j+1}\mu^{-j}\right)\sum_{i=2}\partial_p Q_i(\lambda)\mu^{-i} = \sum_{n=1} up(\lambda)\frac{\lambda^n_{[1]}}{n}\mu^{-n},$$

which is just (6.1). Substituting $p(\mu)$ defined in (5.6) into (6.1) and extracting the equations corresponding to μ^{-i}, we have

$$\begin{aligned} \mu^{-1}:&\quad \frac{\partial p}{\partial\lambda}pu = \frac{1}{u}\frac{\partial Q_2}{\partial\lambda}, \\ \mu^{-n}:&\quad \frac{\partial p}{\partial\lambda}p\frac{\lambda^n_{[1]}}{n} = \frac{1}{u}\frac{\partial Q_{n+1}}{\partial\lambda} - \frac{u_0}{u}\frac{\partial Q_n}{\partial\lambda} - p\frac{\partial Q_n}{\partial\lambda} - \sum_{j=2}^{n-1}\frac{1}{n-j}\frac{\partial v_{n-j}}{\partial x}\frac{\partial Q_j}{\partial\lambda}, \qquad n \geqslant 2. \end{aligned}$$

Now using (5.6) we can rewrite the above equations in terms of $\mathcal{F}_{nm}$ as

$$\mu^{-1}: \quad 2\mathcal{F}_{xx}\mathcal{F}_{0x} + \frac{1}{2}\mathcal{F}_{xx}^2\mathcal{F}_{2x} + \mathcal{F}_{xx}\sum_{i=2}\left(\frac{1}{i}-1\right)\mathcal{F}_{ix}\lambda^{-i} - 2\mathcal{F}_{0x}\sum_{i=1}\mathcal{F}_{ix}\lambda^{-i-1} - \frac{1}{2}\mathcal{F}_{xx}^2\sum_{i=2}\mathcal{F}_{2i}\lambda^{-i-1} - \sum_{i,j=1}\frac{1}{j}\mathcal{F}_{ix}\mathcal{F}_{jx}\lambda^{-i-j-1} = 0,$$

$$\begin{aligned}\mu^{-n}: \quad & \frac{2}{n}\mathcal{F}_{xx}\mathcal{F}_{0x}\mathcal{F}_{nx} + \frac{1}{n+1}\mathcal{F}_{xx}^2\mathcal{F}_{n+1,x} + \mathcal{F}_{xx}\sum_{j=2}^{n-1}\frac{1}{j(n-j)}\mathcal{F}_{n-j,x}\mathcal{F}_{jx} \\ & - \frac{1}{n}\mathcal{F}_{xx}\mathcal{F}_{nx}\sum_{i=1}\mathcal{F}_{ix}\lambda^{-i} - \frac{2}{n}\mathcal{F}_{0x}\mathcal{F}_{nx}\sum_{i=1}\mathcal{F}_{ix}\lambda^{-i-1} \\ & + \frac{1}{n}\mathcal{F}_{xx}^2\sum_{i=2}\mathcal{F}_{ni}\lambda^{-i} - \frac{1}{n+1}\mathcal{F}_{xx}^2\sum_{i=2}\mathcal{F}_{n+1,i}\lambda^{-i-1} + \mathcal{F}_{xx}\sum_{i=1}\frac{1}{i}\mathcal{F}_{ix}\lambda^{n-i-1} \\ & - \mathcal{F}_{x,x}\sum_{j=2}^{n-1}\frac{1}{n-j}\mathcal{F}_{n-j,x}\lambda^{j-1} - \frac{1}{n}\mathcal{F}_{nx}\sum_{i,j=1}\frac{1}{i}\mathcal{F}_{ix}\mathcal{F}_{jx}\lambda^{-i-j-1} \\ & + \frac{1}{n}\mathcal{F}_{xx}\sum_{i=1,j=2}\frac{1}{i}\mathcal{F}_{ix}\mathcal{F}_{nj}\lambda^{-i-j-1} - \mathcal{F}_{xx}\sum_{i=2}\sum_{j=2}^{n-1}\frac{1}{j(n-j)}\mathcal{F}_{n-j,x}\mathcal{F}_{j,i}\lambda^{-i-1} = 0, \\ & n \geqslant 2. \end{aligned} \tag{6.3}$$

We list first several of them in appendix A. In fact, the dispersionless Hirota equations provide a number of relations between the $\mathcal{F}_{nm}$ such that only two of them, say, $\mathcal{F}_{xx}$ and $\mathcal{F}_{2x}$ are independent. For instance,

$$\mathcal{F}_{0x} = -\frac{1}{4}\mathcal{F}_{xx}\mathcal{F}_{2x}, \qquad \mathcal{F}_{3x} = -\frac{3}{\mathcal{F}_{xx}} + \frac{3}{4}\mathcal{F}_{2x}^2, \tag{6.4}$$

$$\mathcal{F}_{22} = \frac{2}{\mathcal{F}_{xx}^2}, \qquad \mathcal{F}_{4x} = -\frac{6\mathcal{F}_{2x}}{\mathcal{F}_{xx}} + \frac{\mathcal{F}_{2x}^3}{2}, \tag{6.5}$$

$$\mathcal{F}_{23} = \frac{3\mathcal{F}_{2x}}{\mathcal{F}_{xx}^2}, \qquad \vdots \tag{6.6}$$

Note that from equations (5.5) and (6.4) we have $u_0 = \mathcal{F}_{2x}/2$, whereas (6.5) together with (6.6) yields

$$\mathcal{F}_{yt} = \tfrac{3}{2}\mathcal{F}_{yy}\mathcal{F}_{xy}$$

which is nothing but the (2 + 1)-dimensional dDym equation (2.2). Similarly, other higher flows can also be extracted from (6.3).

7. Concluding remarks

We have investigated the integrability of the dDym hierarchy through the S function. For finite-dimensional reductions, surprisingly, the dDym hierarchy can be characterized by a single function (free energy) only for a two-reduction model with an extra flow added. The two primary variables of the system can be expressed in terms of second derivatives of the free energy and the hierarchy flows can be rewritten as a set of dispersionless Hirota equations. Our results answer the problem concerning the existence of free energy for the dDym hierarchy in Lax formulation and are consistent with that obtained by the bi-Hamiltonian formulation [10], where the two-reduction model can be interpreted as a topological Landau–Ginzburg model. After identifying the flat coordinates $t^1 = u_0 = \mathcal{F}_{2x}/2$ and $t^2 = -1/u = \mathcal{F}_{xx}$ the free energy $\mathcal{F}$ has the form [10]

$$\mathcal{F} = \tfrac{1}{2}(t^1)^2 t^2 - \tfrac{1}{2}\log t^2$$

which is contained in the classification of solutions of the WDVV equations by Dubrovin [20] for two primary fields. This free energy $\mathcal{F}$ provides many genus-zero correlation functions and would be a good starting point to construct the high-genus expansions of the dDym system (or Whitham hierarchy) in bi-Hamiltonian formulations[21, 36]. We will discuss this issue elsewhere.

Acknowledgments

We would like to thank J H Chang and L P Teo for useful discussions. We also thank the anonymous referees for useful comments and suggestions. This work is supported by the National Science Council of Taiwan under grant no 93-2112-M-194-011.

Appendix. Some equations extracted from (6.3)

In this appendix, we extract dispersionless Hirota equations from (6.3) up to μ^{-4} and λ^{-4}.

$\boxed{\mu^{-1}}$

$$\begin{aligned}
\lambda^0:&\quad 2\mathcal{F}_{0x} + \tfrac{1}{2}\mathcal{F}_{xx}\mathcal{F}_{2x} = 0,\\
\lambda^{-2}:&\quad \tfrac{1}{2}\mathcal{F}_{xx}\mathcal{F}_{2x} + 2\mathcal{F}_{0x} = 0,\\
\lambda^{-3}:&\quad \tfrac{2}{3}\mathcal{F}_{xx}\mathcal{F}_{3x} + 2\mathcal{F}_{0x}\mathcal{F}_{2x} + \tfrac{1}{2}\mathcal{F}_{xx}^2\mathcal{F}_{22} + 1 = 0,\\
\lambda^{-4}:&\quad 2\mathcal{F}_{0x}\mathcal{F}_{3x} + \tfrac{3}{2}\mathcal{F}_{2x} + \tfrac{1}{2}\mathcal{F}_{xx}^2\mathcal{F}_{23} + \tfrac{3}{4}\mathcal{F}_{xx}\mathcal{F}_{4x} = 0,
\end{aligned}$$

$\boxed{\mu^{-2}}$

$$\begin{aligned}
\lambda^0:&\quad \mathcal{F}_{0x}\mathcal{F}_{2x} + \tfrac{1}{3}\mathcal{F}_{xx}\mathcal{F}_{3x} + 1 = 0,\\
\lambda^{-2}:&\quad \tfrac{1}{2}\mathcal{F}_{xx}^2\mathcal{F}_{22} - \tfrac{1}{2}\mathcal{F}_{xx}\mathcal{F}_{2x}^2 - \mathcal{F}_{0x}\mathcal{F}_{2x} + \tfrac{1}{3}\mathcal{F}_{xx}\mathcal{F}_{3x} = 0,\\
\lambda^{-3}:&\quad \tfrac{1}{6}\mathcal{F}_{xx}^2\mathcal{F}_{23} - \tfrac{1}{2}\mathcal{F}_{2x} - \mathcal{F}_{0x}\mathcal{F}_{2x}^2 - \tfrac{1}{2}\mathcal{F}_{xx}\mathcal{F}_{2x}\mathcal{F}_{3x} + \tfrac{1}{4}\mathcal{F}_{xx}\mathcal{F}_{4x} = 0,\\
\lambda^{-4}:&\quad \tfrac{1}{2}\mathcal{F}_{xx}^2\mathcal{F}_{24} - \tfrac{1}{3}\mathcal{F}_{xx}^2\mathcal{F}_{33} + \tfrac{1}{2}\mathcal{F}_{xx}\mathcal{F}_{22} - \tfrac{3}{4}\mathcal{F}_{2x}^2 - \mathcal{F}_{0x}\mathcal{F}_{2x}\mathcal{F}_{3x}\\
&\qquad - \tfrac{1}{2}\mathcal{F}_{2x}\mathcal{F}_{xx}\mathcal{F}_{4x} + \tfrac{1}{5}\mathcal{F}_{xx}\mathcal{F}_{5x} = 0,
\end{aligned}$$

$\boxed{\mu^{-3}}$

$$\begin{aligned}
\lambda^0:&\quad \tfrac{2}{3}\mathcal{F}_{0x}\mathcal{F}_{3x} + \tfrac{1}{4}\mathcal{F}_{xx}\mathcal{F}_{4x} + \mathcal{F}_{2x} = 0,\\
\lambda^{-2}:&\quad \tfrac{1}{3}\mathcal{F}_{xx}^2\mathcal{F}_{32} - \tfrac{1}{3}\mathcal{F}_{xx}\mathcal{F}_{2x}\mathcal{F}_{3x} - \tfrac{2}{3}\mathcal{F}_{0x}\mathcal{F}_{3x} + \tfrac{1}{4}\mathcal{F}_{xx}\mathcal{F}_{4x} = 0,\\
\lambda^{-3}:&\quad \tfrac{1}{3}\mathcal{F}_{xx}^2\mathcal{F}_{33} - \tfrac{1}{4}\mathcal{F}_{xx}^2\mathcal{F}_{42} - \tfrac{1}{3}\mathcal{F}_{xx}\mathcal{F}_{3x}^2 - \tfrac{2}{3}\mathcal{F}_{3x}\mathcal{F}_{0x}\mathcal{F}_{2x} - \tfrac{1}{3}\mathcal{F}_{3x}\\
&\qquad + \tfrac{1}{5}\mathcal{F}_{xx}\mathcal{F}_{5x} - \tfrac{1}{2}\mathcal{F}_{xx}\mathcal{F}_{22} = 0,\\
\lambda^{-4}:&\quad \tfrac{1}{12}\mathcal{F}_{xx}^2\mathcal{F}_{34} - \tfrac{2}{3}\mathcal{F}_{0x}\mathcal{F}_{3x}^2 - \tfrac{1}{2}\mathcal{F}_{3x}\mathcal{F}_{2x} - \tfrac{1}{3}\mathcal{F}_{xx}\mathcal{F}_{4x}\mathcal{F}_{3x} + \tfrac{1}{6}\mathcal{F}_{xx}\mathcal{F}_{6x}\\
&\qquad - \tfrac{1}{6}\mathcal{F}_{xx}\mathcal{F}_{23} = 0,
\end{aligned}$$

$\boxed{\mu^{-4}}$

$$\begin{aligned}
\lambda^0:&\quad \tfrac{1}{2}\mathcal{F}_{0x}\mathcal{F}_{4x}+\tfrac{1}{5}\mathcal{F}_{xx}\mathcal{F}_{5x}+\tfrac{2}{3}\mathcal{F}_{3x}+\tfrac{1}{4}\mathcal{F}_{2x}^2=0,\\
\lambda^{-2}:&\quad \tfrac{1}{4}\mathcal{F}_{xx}^2\mathcal{F}_{42}-\tfrac{1}{4}\mathcal{F}_{xx}\mathcal{F}_{2x}\mathcal{F}_{4x}-\tfrac{1}{2}\mathcal{F}_{0x}\mathcal{F}_{4x}+\tfrac{1}{5}\mathcal{F}_{xx}\mathcal{F}_{5x}=0,\\
\lambda^{-3}:&\quad \tfrac{1}{4}\mathcal{F}_{xx}^2\mathcal{F}_{43}-\tfrac{1}{5}\mathcal{F}_{xx}^2\mathcal{F}_{52}-\tfrac{1}{4}\mathcal{F}_{xx}\mathcal{F}_{3x}\mathcal{F}_{4x}-\tfrac{1}{2}\mathcal{F}_{4x}\mathcal{F}_{0x}\mathcal{F}_{2x}\\
&\qquad -\tfrac{1}{4}\mathcal{F}_{4x}-\tfrac{1}{4}\mathcal{F}_{xx}\mathcal{F}_{2x}\mathcal{F}_{22}-\tfrac{1}{3}\mathcal{F}_{xx}\mathcal{F}_{32}+\tfrac{1}{6}\mathcal{F}_{xx}\mathcal{F}_{6x}=0,\\
\lambda^{-4}:&\quad \tfrac{1}{4}\mathcal{F}_{xx}^2\mathcal{F}_{44}-\tfrac{1}{5}\mathcal{F}_{xx}^2\mathcal{F}_{53}-\tfrac{1}{2}\mathcal{F}_{4x}\mathcal{F}_{0x}\mathcal{F}_{3x}-\tfrac{3}{8}\mathcal{F}_{2x}\mathcal{F}_{4x}-\tfrac{1}{4}\mathcal{F}_{xx}\mathcal{F}_{4x}^2\\
&\qquad +\tfrac{1}{7}\mathcal{F}_{xx}\mathcal{F}_{7x}+\tfrac{1}{4}\mathcal{F}_{xx}\mathcal{F}_{42}-\tfrac{1}{4}\mathcal{F}_{xx}\mathcal{F}_{2x}\mathcal{F}_{23}-\tfrac{1}{3}\mathcal{F}_{xx}\mathcal{F}_{33}=0.
\end{aligned}$$

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